Limits

Recall from Calculus I:

$f(x)$ has limit as $x \to a$ if

1. $\lim\limits_{x \to a^-}$ exists.
2. $\lim\limits_{x \to a^+}$ exists
3. They are equal.

Calculus III:

$f(x,y)$ has a limit as $(x,y) \to (a,b)$ if you can walk up to (a,b) from any direction in the xy-plane (the input) & you get the same value for the limit of the $f(x,y)$ (the output)

Bad news: Very tricky

Ex) $f(x,y) = \dfrac{2x+3y}{x+y}$

lets walk upto $(0,0)$ along the positive x-axis.

(so $y=0$) we get $\lim\limits_{x\text{-axis}^+} = 2$

Along the positive y-axis:

($x=0$)

$$\lim_{y\text{-axis}^+} = 3$$

So the limit doesn't exist.

Properties of limits of functions of two variables.

The following rules hold if L, M, & k are real numbers &

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

$$\& \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

1. Sum Rule: $\lim\limits_{(x,y)\to(x_0,y_0)} \left(f(x,y) \pm g(x,y)\right) = L \pm M$

2. $\lim\limits_{(\rightarrow)} f(x,y)\cdot g(x,y) = L\cdot M$

3. $\lim \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$

4. $k\,f(x,y) = kL$

Ex) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}\left(\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$

$\lim \dfrac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{x-y}}$

$\lim x(\sqrt{x}+\sqrt{y}) = 0(0+0) = 0$

Ex) $\lim\limits_{(x,y)\to(0,1)} \dfrac{x-xy+3}{x^2y+5xy-y^3} = \dfrac{0-0\cdot 1+3}{0+0+(-1)}$

$= -3$

Ex] $\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{3^2+(-4)^2}$

$= \sqrt{9+16} = 5$

Ex] $\lim_{P\to(1,0,-1)} \dfrac{e^{x+z}}{z^2+\cos\sqrt{xy}} = \dfrac{e^0}{(-1)^2+\cos(0)} = \dfrac{1}{2}$

Ex] $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

$x^2+y^2=r^2$

$\left.\begin{array}{l} x\to 0 \\ y\to 0 \end{array}\right\} \to r\to 0$

$= \lim_{r\to 0} \dfrac{\sin r^2}{r^2} = \dfrac{0}{0}$

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$= \lim_{r\to 0} \dfrac{\cos(r^2)(2\cancel{r})}{2\cancel{r}} = 1$

Ex] $\lim_{(x,y)\to(1,-1)} \dfrac{x^3+y^3}{x+y}$

$(x,y)\to(1,1)$

$$= \lim \frac{\cancel{(x+y)}(x^2-xy+y^2)}{\cancel{(x+y)}} = \lim x^2-xy+y^2$$

$$= 1+1+1=3$$

long division

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

$$
\begin{array}{r l}
 & x^2+y^2-xy \\
x+y & \overline{)\,x^3+0x^2+0x+y^3+0y^2+0y} \\
 & -(x^3+x^2y) \\
\hline
 & -x^2y+y^3 \\
 & -(+y^3+y^2x) \\
\hline
 & -x^2y-y^2x \\
 & -(-x^2y-y^2x) \\
\hline
 & 0
\end{array}
$$

Ex) $\lim_{P\to(\pi,0,2)} z\,e^{-2y}\cos 2x$

$p \to (1,0,1)$

$= 3e^0 \cos(2\pi) = 3 \cdot 1 \cdot (1) = 3$